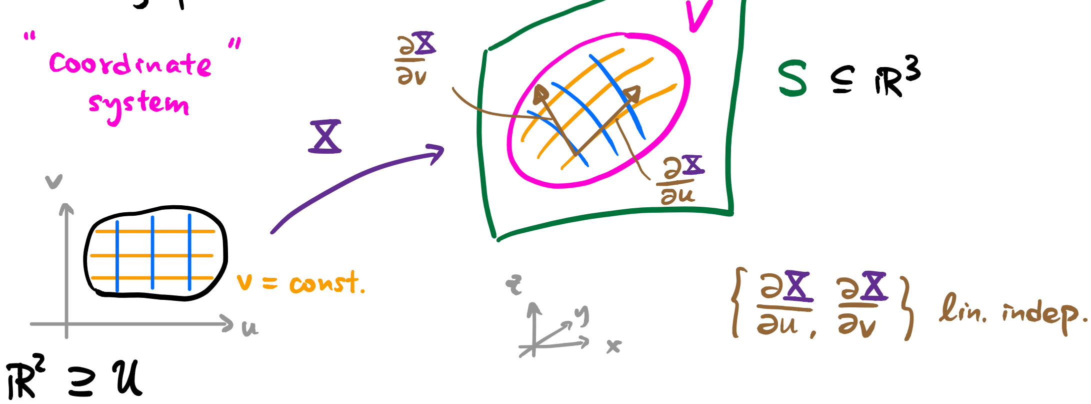
§ Differential Calculus on surfaces

Recall that a **surface** is a subset $S \subseteq \mathbb{R}^3$ covered by parametrizations:



Examples: Spheres, torus, graphs $z = f(x,y)$ ......

Goal: Do calculus on surfaces.

Let's recall two important theorems from multivariable calculus.

Inverse Function Theorem

Let $F : \mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth map. $p_0 \in \mathcal{U}$.

Suppose $dF|_{p_0} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism.

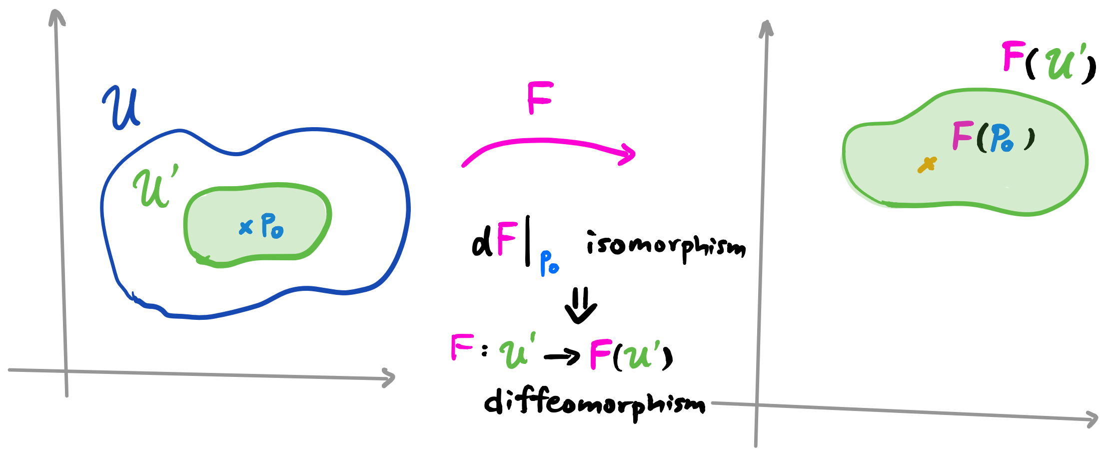
Then, F is a local diffeomorphism near p_0.

i.e. $\exists$ nbd $\mathcal{U}' \subseteq \mathcal{U}$ of p_0 s.t.

$$F|_{\mathcal{U}'} : \mathcal{U}' \longrightarrow F(\mathcal{U}')$$

is a diffeomorphism, i.e. smooth, bijective with smooth inverse.

Inverse function theorem



Implicit Function Theorem

Let $F = F(x,y,z) : O \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function.

Consider the level surface of F at $a \in \mathbb{R}$

$$F^{-1}(a) := \{ p \in O : F(p) = a \}.$$

Suppose $p_0 = (x_0, y_0, z_0) \in F^{-1}(a)$ and $\boxed{\left.\frac{\partial F}{\partial z}\right|_{p_0} \neq 0}$.

Then, $\exists$ a nbd $V \subseteq \mathbb{R}^3$ of p_0 & a smooth function

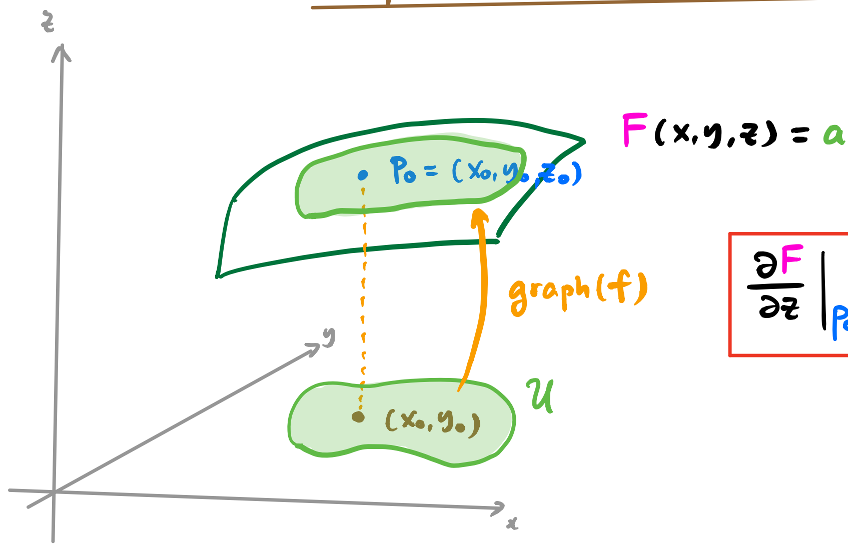
$$f : \underset{\text{open}}{\mathcal{U} \subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}$$

$$(x_0, y_0) \longmapsto z_0$$

s.t. $F^{-1}(a) \cap V = \text{graph}(f)$

$$= \{ (x, y, f(x,y)) : (x,y) \in \mathcal{U} \}.$$

Implicit function theorem



$$\boxed{\left.\frac{\partial F}{\partial z}\right|_{P_0} \neq 0}$$

Proposition: Any surface $S \subseteq \mathbb{R}^3$ is locally a graph,

i.e. for each $p \in S$, $\exists$ nbd V of p in S s.t.

$$V = \{ z = f(x,y) \} \text{ or } \{ y = f(x,z) \} \text{ or } \{ x = f(y,z) \}$$

for some smooth function $f : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Proof: Fix $p \in S$, $\exists$ parametrization

$$\begin{array}{rcl} \mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 & \xrightarrow{\cong} & V \subset S \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ q & \longmapsto & p \end{array}, \quad \mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v))$$

s.t.

$$d\mathbf{X}\Big|_q = \left.\begin{pmatrix} \boxed{\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}} \\ \begin{matrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{matrix} \end{pmatrix}\right|_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

is 1-1

($\Longleftrightarrow$ rank = 2)

($\Longleftrightarrow$ $\exists$ 2×2 invertible submatrix)

By Inverse Function Theorem, locally we can solve u, v in terms of x, y, i.e.

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases} \Rightarrow$$ locally near p in S is the graph of the function

$$z(u(x,y), v(x,y))$$

$\square$

Given a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$, for each $a \in \mathbb{R}$, consider the level set

$$F^{-1}(a) := \{ p \in \mathbb{R}^3 : F(p) = a \}$$

Question: When is it a surface?

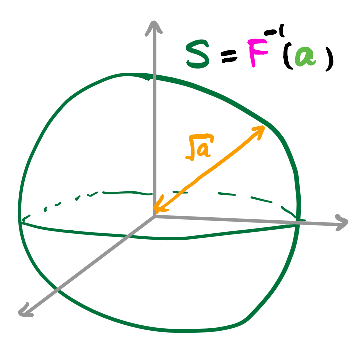
Def$^{\underline{n}}$: a is a regular value of F if $\forall p \in F^{-1}(a)$, $\nabla F|_p \neq 0$.

Theorem: $F^{-1}(a)$ is a surface for any regular value a of F.

Proof: The implicit function theorem implies that $F^{-1}(a)$ is locally a graph, hence it is a surface. $\square$

Example: $F(x,y,z) = x^2 + y^2 + z^2$

$F^{-1}(a)$ = Sphere of radius $\sqrt{a}$



"Singular" when $a = 0$ since

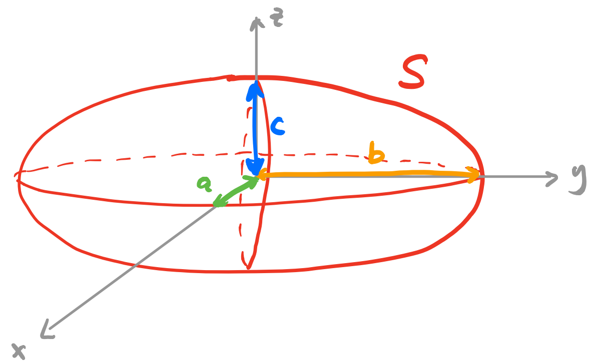
$$\nabla F(0) = (2x, 2y, 2z)\Big|_0 = 0.$$

Example: Ellipsoid

Let $a, b, c > 0$ be constants.

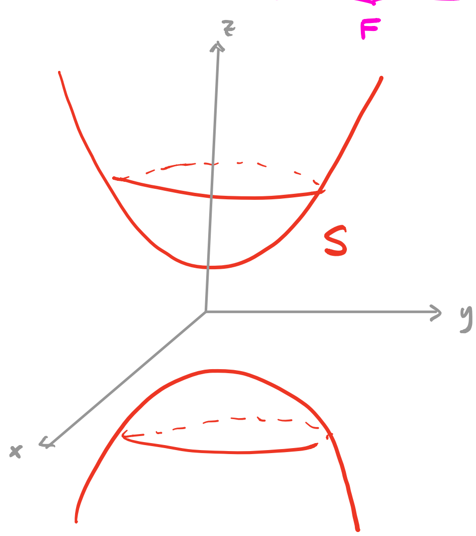
$$S = \left\{ (x,y,z) : \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}_{F(x,y,z) :=} = 1 \right\} = F^{-1}(1)$$

Then, $\nabla F = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right) \neq 0 \quad \forall p \in S.$



Example: 2-sheeted hyperboloid

$$S = \{ (x,y,z) : \underbrace{-x^2 - y^2 + z^2}_{F} = 1 \}$$



$\nabla F = (-2x, -2y, 2z)$

$= \vec{0}$ only at origin.